SERGE LANG'S ALGEBRA CHAPTER 3 EXERCISE SOLUTIONS

KELLER VANDEBOGERT

1. Problem 1

Let $\{u_1, \dots, u_n\}$ and $\{w_1, \dots, w_m\}$ be bases for U and W, respectively. Without loss of generality, we may assume that $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$ form bases for $U \cap W$. This implies

$$\operatorname{Span}\{u_1, \dots, u_k\} = \operatorname{Span}\{w_1, \dots, w_k\}$$

so that $\{u_1, \dots, u_n, w_{k+1}, \dots, w_m\}$ forms a basis for $U + W$. Counting cardinalities,

$$\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$$
$$\implies \dim(U+W) + \dim(U \cap W) = \dim(U) + \dim(W)$$

2. Problem 2

As $S^{-1}M \cong S^{-1}R \otimes_R M$, we may assume without loss of generality that R is local. Let $\mathfrak{m}$ denote the maximal ideal; $R/\mathfrak{m}$ is a field, so that $R/\mathfrak{m} \otimes_R M$ is a vector space.

Choosing a basis yields a generating set for the preimage, and conversely, every generating set can be refined to a basis in $R/\mathfrak{m} \otimes_R M$. Since vector spaces have the invariant basis property, we deduce that M does as well.

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3. Problem 3

Since R is an integral domain, the homothety map $s \mapsto rs$ is injective. Extending by linearity over k, we have an injective map over a finite dimensional vector space. But this means that we have an isomorphism, thus there exists $s \in R$ such that $rs = 1$, whence R is a field.

4. Problem 4

(a). Assume first that

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

splits, so that $M \cong M' \oplus M''$. Then, take $h : M \oplus M'' \to M'$ as the natural projection onto M'. By definition, $hf \equiv \mathrm{id}$.

Similarly, we may take $i : M'' \hookrightarrow M' \oplus M''$ to be the standard inclusion. Again by definition, we have that $ig \equiv \mathrm{id}$.

Assume no that f is left invertible, with left inverse h. Observe first that for any $m \in M$,

$$h\big(m - f \circ h(m)\big) = h(m) - h(m) = 0$$

So that $m - f(h(m)) \in \operatorname{Ker} k$. This immediately gives that $M = \operatorname{Ker} h + \operatorname{Im} f$, since $m = (m - f(h(m))) + f(h(m))$. Indeed, we can say even more, since if $m \in \operatorname{Ker} h \cap \operatorname{Im} f$, then $m = f(m')$ for some $m' \in M'$, and

$$0 = h(m) = h(f(m')) = m'$$

So $m' = 0 \implies f(m') = 0$. Hence

$$M = \operatorname{Ker} h \oplus \operatorname{Im} f$$

Since f is injective, $M' \cong \operatorname{Im} f$. It remains to show that

$$\operatorname{Ker} h \cong M''$$

Since g is surjective, every $m'' \in M''$ is of the form $g(m)$ for some $m \in M$. As $M = \operatorname{Ker} h \oplus \operatorname{Im} f$ and $\operatorname{Ker} g = \operatorname{Im} f$ by exactness,

$$g(M) = g(\operatorname{Ker} h) = M''$$

And, as $g|_{\operatorname{Ker} h}$ is injective by exactness,

$$\operatorname{Ker} h \cong M''$$

so that $M \cong M' \oplus M''$.

Suppose now that g is right invertible with right inverse i. Consider

$$m - i(g(m))$$

We again see that

$$g\big(m - i(g(m))\big) = g(m) - g(m) = 0$$

So that $m - i(g(m)) \in \operatorname{Ker} g = \operatorname{Im} f$ (by exactness). Also, if $m \in \operatorname{Ker} g \cap \operatorname{Im} i$, then $m = i(m'')$ for some $m'' \in M''$ and

$$0 = g(m) = g(i(m'')) = m''$$

So that $m = 0$ as well. Hence

$$M \cong \operatorname{Ker} g \oplus \operatorname{Im} i$$

Since $\operatorname{Ker} g = \operatorname{Im} f$ and f is injective, $\operatorname{Im} f = M'$. Similarly, we deduce that

$$g(M) = g(\operatorname{Im} i) = M''$$

and by the first isomorphism theorem, this must be an isomorphism. Hence,

$$M \cong M' \oplus M''$$

as desired.

(b). Let

$$i : E \to \bigoplus_i E_i$$

$$\pi : \bigoplus_i E_i \to E$$

be the given maps. These are trivially R-module homomorphisms. We also see

$$\begin{aligned}
\pi \circ i(x) &= \pi(\psi_1 x, \dots, \psi_m x) \\
&= \phi_1 \circ \psi_1 x + \cdots + \phi_m \circ \psi_m x \\
&= \Big(\sum_i \phi_i \psi_i \Big)(x) \\
&= x
\end{aligned}$$

$$\begin{aligned}
i \circ \pi(x_1, \dots, x_m) &= i(\phi_1 x_1 + \cdots + \phi_m x_m) \\
&= \psi_1 \phi_1 x_1, \dots, \psi_m \phi_m x_m) \\
&= (x_1, \dots, x_m)
\end{aligned}$$

Whence i and π are inverses of each other, so they are isomorphisms.

If each ϕ_i is the natural inclusion $E_i \hookrightarrow \bigoplus_i E_i$ and ψ_i the natural projection $\bigoplus_i E_i \to e_i$, we see

$$\psi_i \phi_i = \mathrm{id}, \quad \psi_j \circ \phi_i \equiv 0 \ (i \neq j)$$

We also see for $(e_1, \dots, e_m) \in \bigoplus_i E_i$,

$$\begin{aligned}
&\phi_1 \psi_1 (e_1, \dots, e_m) + \cdots + \phi_m \psi_m (e_1, \dots, e_m) \\
=&\phi_1(e_1) + \cdots + \phi_m(e_m) \\
=&(e_1, 0, \dots, 0) + \cdots + (0, \dots, 0, e_m) \\
=&(e_1, \dots, e_m)
\end{aligned}$$

So that $\sum_i \phi_i \psi_i = \mathrm{id}$, as required.

5. Problem 5

Proceed by induction on the maximal amount of linearly independent element of A over $\mathbb{R}$. For the base case $n = 1$, this is by definition.

Let $\{v_1, \dots, v_m\}$ be a maximal set of such elements. Consider any subgroup A_0 contained in the space generated by $\{v_1, \dots, v_{m-1}\}$. By the inductive hypothesis, these can all be generated by integral linear combinations.

Now, denote by S the subset of A such that

$$v = a_1v_1 + \cdots + a_mv_m,\ \ a_i \in \mathbb{R}$$

$$0 \leqslant a_i < 1$$

$$0 \leqslant a_m \leqslant m$$

Choose v_m' such that the coefficient a_m is minimal and nonzero in S. Note that such an element exists since S is finite by assumption and if every $a_m = 0$, $\{v_1, \dots, v_{m-1}\}$ generates S and by scaling, $\{v_1, \dots, v_{m-1}\}$ generates A. Employing the inductive hypothesis would yield the result, whence we may find $a_m > 0$.

We want to now show that

$$\{v_1, \dots, v_{m-1}, v_m'\}$$

is a basis for A over $\mathbb{Z}$. Let $v \in A$, so that $v = a_1v_1 + \cdots + a_mv_m$. Then we may find a sufficiently large $N \in \mathbb{N}$ such that $v/N \in S$; by definition, $a_m/N \geqslant a_m'$, where a_m' is the mth coefficient of v_m'.

Let k be the smallest positive integer such that $ka_m' \geqslant a_m/N$. If $ka_m' \neq a_m/N$, then by minimality of k,

$$\frac{a_m}{N} - ka_m' < a_m'$$

But this may not happen, so in fact

$$ka'_m = \frac{a_m}{N}$$

and if

$$v'_m = a'_1 v_1 + \cdots + a'_m v_m$$

for some coefficients a'_i, we may multiply by the above by Nk and use that $Nka'_m = a_m$ and see

$$a_m v_m = -Nka'_1 v_1 - \cdots - Nka'_{m-1} + Nkv'_m$$

And, substituting this for the expression of v,

$$v = (a_1 - Nka'_1)v_1 + \cdots + (a_{m-1} - Nka'_{m-1})v_{m-1} + Nkv'_m$$

Subtracting we see that $v - Nkv'_m \in A_0$, so by the inductive hypothesis we may find $j_i \in \mathbb{Z}$ such that

$$v - Nkv'_m = j_1 v_1 + \dots j_{m-1} v_{j-1}$$

Whence we finally see $v \in \text{Span}_{\mathbb{Z}}\{v_1, \dots, v'_m\}$, as desired.

6. Problem 6

Confer Lang's Algebraic topology book for the correct statement. The statement given here is not true.

7. Problem 7

(a). Let u, $v \in W$. Then,

$$|u - v| \leqslant |u| + |v| = 0$$
$$\implies |u - v| = 0$$

So this is a subgroup.

(b). For convenience, we may assume $M_0 = \{0\}$. Let $M_1 = (v_1, \ldots, v_r)$ and let $d \in \mathrm{Ann}(M/M_1)$. Then, $dM \subset M_1$, and we may choose $n_{j,j}$ to be the smallest integer such that there exist

$$n_{j,1}, \ldots, n_{j,j-1} \in \mathbb{Z}$$

such that

$$n_{j,1}v_1 + \cdots + n_{j,j}v_j = dw_j$$

for some $w_j \in M$. Without loss of generality, we may assume $0 \leqslant n_{j,k} \leqslant d-1$. It remains to show our elements $\{w_1, \ldots, w_r\}$ forms a basis.

By selection

$$\mathrm{Span}\{w_1, \ldots, w_r\} = \mathrm{Span}\{v_1, \ldots, v_r\}$$

And, since the cardinality of the w_i matches that of the v_i, linear independence is a triviality. Finally, since $0 \leqslant n_{j,k} \leqslant d-1$,

$$\begin{aligned} |w_i| &= \Big| \sum_{j=1}^{r} \frac{n_{j,k}}{d} v_j \Big| \\ &\leqslant \sum_{j=1}^{r} |v_j| \end{aligned}$$

As desired.

8. Problem 8

(a). We certainly have that the kernel is ± 1. Let $(a, b) = (a', b') = 1$. If $x = a/b$, $y = a'/b'$,

$$\begin{aligned} h(xy) &= \log \max \big(|a||a'|, |b||b'|\big) \\ &\leqslant \log \Big(\max \big(|a|, |b|\big) \max \big(|a'|, |b'|\big)\Big) \\ &= h(x) + h(y) \end{aligned}$$

(b). M is certainly finitely generated, since if not, we could find an infinite irredundant generating set for M_1, which is a contradiction.

Using Problem 7, after completing the x_i to a basis for M, we may bound any generating set appropriately.

9. Problem 9

(a). Define our localization as equivalence classes

$$\frac{m}{s} = \frac{m'}{s'} \iff \exists\, r \in A \text{ such that } r(s'm - sm') = 0$$

This is given the trivial $S^{-1}A$-module structure

$$\frac{a}{b}\left(\frac{m}{s}\right) := \frac{am}{bs}$$

Well definedness/distributeivity follow immediately from the fact that M is itself an A-module.

(b). Let

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

We exact. Then, ϕ and ψ extend to localized maps by defining

$$\phi\left(\frac{m'}{s}\right) := \frac{\phi(m')}{s}$$

$$\psi\left(\frac{m}{s}\right) := \frac{\psi(m)}{s}$$

And then extending by linearity. Suppose then that $\phi(m')/s = 0$, so that some $t \in A$ must annihilate $\phi(m')$, implying that $\phi(tm') = 0$.

Since ϕ is injective, $tm' = 0$, whence $m'/s = 0$, so the localized ϕ is also injective.

Now let us check exactness at $S^{-1}M$. We have $\operatorname{Im}\phi \subset \operatorname{Ker}\psi$ by definition. Suppose that $\psi(m/s) = 0$, so there exists $t \in A$ such that

$\psi(tm) = 0$. By exactness, $tm \in \operatorname{Im}\phi$, so $\phi(m') = tm$ for some $m' \in M'$. Then,

$$\begin{aligned}\frac{m}{s} &= \frac{tm}{ts} \\ &= \frac{\phi(m')}{ts} \\ &= \phi\left(\frac{m}{ts}\right) \in \operatorname{Im}\phi\end{aligned}$$

So our sequence is exact at M. It remains to show that ψ is surjective. Observe that ψ on M is surjective, so given $m'' \in M''$, there exists $m \in M$ such that $\psi(m) = m''$. Then,

$$\frac{m''}{s} = \frac{\psi(m)}{s} = \psi\left(\frac{m}{s}\right) \in \operatorname{Im}\psi$$

So that

$$0 \longrightarrow S^{-1}M' \xrightarrow{\phi} S^{-1}M \xrightarrow{\psi} S^{-1}M'' \longrightarrow 0$$

is exact.

10. Problem 10

(a). Our map is

$$\begin{aligned}M &\to \prod_{\mathfrak{p}\in \text{m-Spec}(A)} M_{\mathfrak{p}} \\ m &\mapsto \left(\frac{m}{1}\right)_{\mathfrak{p}\in\text{m-Spec}(A)}\end{aligned}$$

Now, suppose $m \mapsto (0)$. Then for each $\mathfrak{p} \in \text{m-Spec}(A)$, there exists $a_{\mathfrak{p}} \notin \mathfrak{p}$ such that $a_{\mathfrak{p}} m = 0$. But then $\operatorname{Ann}(m)$ is not contained in any maximal ideal, whence $\operatorname{Ann}(m) = A$, so $m = 0$, yielding surjectivity.

(b). We already know the forward direction from part (b) of the previous problem. Let ϕ, ψ be our maps $\phi : M' \to M$, $\psi : M \to M''$, and consider the converse.

Firstly, suppose $\phi(m') = 0$ for some $m' \in M'$. Then for all $\mathfrak{p} \in$ m-Spec(A), there exists $a_{\mathfrak{p}} \notin \mathfrak{p}$ such that $a_{\mathfrak{p}} m' = 0$. To see this, note that

$$\begin{aligned} \phi(m') = 0 &\implies \phi\Big(\frac{m'}{1}\Big) \\ &\implies \frac{m'}{1} = 0 \text{ for all } \mathfrak{p} \in \text{m-Spec}(A) \end{aligned}$$

By identical reasoning as in part (a), $\operatorname{Ann}(m') = A$, so that $m' = 0$, and ϕ is injective.

We know that $\operatorname{Ker}\psi \subset \operatorname{Im}\phi$. For the reverse inclusion, note that

$$\begin{aligned} &\psi \circ \phi\Big(\frac{m'}{1}\Big) = 0 \\ \implies &\frac{\psi \circ \phi(m')}{1} = 0 \text{ for all } \mathfrak{p} \in \text{m-Spec}(A) \\ \implies &\operatorname{Ann}(\psi \circ \phi(m')) = A \end{aligned}$$

whence $\operatorname{Im}\phi \subset \operatorname{Ker}\psi$, giving exactness at M. Finally, surjectivity is a tautology, so that

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M'' \longrightarrow 0$$

is exact.

(c). Suppose that $M \to M_{\mathfrak{p}}$ and $m \mapsto m/1 = 0$. Then, there exists $a_{\mathfrak{p}} \notin \mathfrak{p}$ such that $a_{\mathfrak{p}} m = 0$. Since M is torsion free, $m \neq 0$ implies that $a_{\mathfrak{p}} = 0$. But then $a_{\mathfrak{p}} \in \mathfrak{p}$, which cannot happen, so we deduce that $m = 0$ and our natural inclusion is thus injective, as desired.

11. Problem 11

Let $\mathfrak{p} \in$ m-Spec$(\mathfrak{o})$. Then $M_{\mathfrak{p}}$ is still finitely generated and torsion free over $\mathfrak{o}_{\mathfrak{p}}$. By problems of the previous chapter, $\mathfrak{o}_{\mathfrak{p}}$ is a PID, and hence $M_{\mathfrak{p}}$ is projective (since it is free). Let

$$F \xrightarrow{f} M \longrightarrow 0$$

be exact. We want to show that f is right invertible. We know that the induced map $f_{\mathfrak{p}}$ is right invertible by freeness of $M_{\mathfrak{p}}$.

Hence for all $\mathfrak{p} \in$ m-Spec($\mathfrak{o}$), there exists $g_{\mathfrak{p}}$ such that $f_{\mathfrak{p}} g_{\mathfrak{p}} = \mathrm{id}_{M_{\mathfrak{p}}}$. We can then find $c_{\mathfrak{p}} \in \mathfrak{o}$ with $c_{\mathfrak{p}} \notin \mathfrak{p}$ and

$$c_{\mathfrak{p}} g_{\mathfrak{p}}(M) \subset F$$

(this is merely by definition), since $g_{\mathfrak{p}}(M_{\mathfrak{p}}) \subset F_{\mathfrak{p}}$. Then, we want to show that $\{c_{\mathfrak{p}}\}_{\mathfrak{p} \in \mathrm{Spec}(\mathfrak{o})}$ generate all of $\mathfrak{o}$; this follows since $\{c_{\mathfrak{m}}\}_{\mathfrak{m} \in \text{m-Spec}(\mathfrak{o})}$ is not contained in any maximal ideal, hence generates all of $\mathfrak{o}$. Thus there exist $x_i,\ c_{\mathfrak{p}_i} \in \mathfrak{o}$ such that

$$\sum_i x_i c_{\mathfrak{p}_i} = 1$$

Set $g := \sum_i x_i c_{\mathfrak{p}_i} g_{\mathfrak{p}_i}$. Then for all $a/b \in \mathfrak{o}$, $\mathfrak{m} \in$ m-Spec($\mathfrak{o}$):

$$\begin{aligned}
(f \circ g)_{\mathfrak{m}}\left(\frac{a}{b}\right) &= \frac{1}{b} \sum_i f_{\mathfrak{m}} \circ (x_i c_{\mathfrak{p}_i} g_{\mathfrak{p}_i}(a)) \\
&= \frac{1}{b} \sum_i x_i c_{\mathfrak{p}_i} f_{\mathfrak{p}_i} g_{\mathfrak{p}_i}(a) \\
&= \frac{a}{b} \sum_i x_i c_{\mathfrak{p}_i} = \frac{a}{b}
\end{aligned}$$

Since the maximal ideal $\mathfrak{m}$ was arbitrary, we deduce that $f \circ g \equiv \mathrm{id}_M$.

12. Problem 12

(a). We have the following short exact sequence

$$0 \longrightarrow \mathfrak{a} \cap \mathfrak{b} \longrightarrow \mathfrak{a} \oplus \mathfrak{b} \longrightarrow \mathfrak{a} + \mathfrak{b} \longrightarrow 0$$

Assume then that $\mathfrak{a}$ and $\mathfrak{b}$ are coprime, so that $\mathfrak{a} \cap \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, $\mathfrak{a} + \mathfrak{b} = \mathfrak{o}$. As $\mathfrak{o}$ is projective, the sequence above splits, so

$$\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{o} \oplus \mathfrak{a}\mathfrak{b}$$

Now, employing exercise 19 of the previous chapter, choose x, $y \in \mathfrak{o}$ such that $x\mathfrak{a}$ and $y\mathfrak{b}$ are coprime. Then,

$$\begin{aligned}\mathfrak{a} \oplus \mathfrak{b} &= x\mathfrak{a} \oplus y\mathfrak{b} \\ &= \mathfrak{o} \oplus xy\mathfrak{a}\mathfrak{b} \\ &= \mathfrak{o} \oplus \mathfrak{a}\mathfrak{b}\end{aligned}$$

Whence the general case. Indeed, by induction, one easily sees

$$\mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n = \mathfrak{o}^{n-1} \oplus \mathfrak{a}_1\mathfrak{a}_2 \ldots \mathfrak{a}_n$$

(b). Let $f : \mathfrak{a} \to \mathfrak{b}$ be our isomorphism. Then, $f_k|_{\mathfrak{a}} = f$, and

$$f_k(a) = f_k(1) \cdot a = ca$$

for each $a \in \mathfrak{a}$. Hence, f is merely the homothety $m_c : x \mapsto cx$, where $c := f_k(1)$.

(c). Let $f \in \mathrm{Hom}(\mathfrak{a}, \mathfrak{o})$. Certainly $1 \notin \mathfrak{a}$, and we may extend f to all of k by linearity as in (b). Then the association

$$f \mapsto f_k(1) \in \mathfrak{a}^{-1}$$

is an isomorphism. Note that well definedness follows since if $f(a) \in \mathfrak{o}$ for $a \in \mathfrak{a}$, then $f_k(1) \cdot a \in \mathfrak{o}$, so that $f_k(1) \in \mathfrak{a}^{-1}$ by definition.

Injectivity is easy since if $f_k(1) = g_k(1)$, then for all $a \in \mathfrak{a}$, $f_k(1) \cdot a = g_k(1) \cdot a \implies f(a) = g(a)$, whence $f \equiv g$. Surjectivity follows from part (b), so this is indeed an isomorphism.

In particular,

$$\mathrm{Hom}(\mathfrak{a}, \mathfrak{o}) = \mathfrak{a}^{-1}$$

and

$$\mathfrak{a}^{\vee\vee} = (\mathfrak{a}^{-1})^{-1} = \mathfrak{a} \implies \mathfrak{a}^{\vee\vee} = \mathfrak{a}$$

13. Problem 13

M is projective, hence a direct summand of a free module F. This immediately gives that M is torsion free, so the free module F' generated by the non torsion elements is contained in M. By definition (since we have only removed torsion elements) the rank of F ad F' must coincide, since else F' would have nontrivial torsion. Thus, there exists F and F' free such that

$$F' \subset M \subset F, \quad \operatorname{rank} F = \operatorname{rank} F'$$

(b). Proceed by induction on the rank of M. When $n = 1$, there is nothing to prove.

Assume now that M has rank n. Choose generators $e_1, \ldots, e_{n-1}$ linearly independent with span denoted by N. We have the short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

By the inductive hypothesis, $N = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_{n-1}$. Also, counting ranks yields that M/N has rank 1, whence we may choose a generator of M/N. Its preimage will by linearly independent with N since it has nonzero class in M/N, in which case we see that $M \cong \bigoplus_i \mathfrak{a}_i$.

As $\mathfrak{o}$ is Noetherian, $\mathfrak{a}_n$ is finitely generated and M/N is torsion free/ Thus M/N is projective, so our sequence splits

$$\implies M = \mathfrak{a}_1 \oplus \cdots \oplus \mathfrak{a}_n$$

As desired.

(c). We may assume without loss of generality that the $\mathfrak{a}_i$ are pairwise coprime. By Problem 12 part (a),

$$M = \mathfrak{o}^{n-1} \oplus \mathfrak{a}$$

where $\mathfrak{a} = \mathfrak{a}_1 \dots \mathfrak{a}_n$. Suppose now that for any two fractional ideals $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{o}$, that $\mathfrak{o}^{n-1} \oplus \mathfrak{a} = \mathfrak{o}^{m-1} \oplus \mathfrak{b}$. We want to show that this is possible if and only if $\mathfrak{a} = \mathfrak{b}$ and $n = m$.

If $\mathfrak{o}^{n-1} \oplus \mathfrak{a} = \mathfrak{o}^{m-1} \oplus \mathfrak{b}$, taking the rank of both sides immediately yields $m = n$. If we take the $(n+1)$th exterior power, we find that

$$D_1 \mathfrak{o}^{n-1} \otimes \mathfrak{a} = D_2 \mathfrak{o}^{n-1} \otimes \mathfrak{b}, \quad D_i \in \mathfrak{o}$$

$$\implies \mathfrak{a} = \mathfrak{b}$$

where we have used the fact that the exterior power converts our direct sum to a tensor product (the D_i are our determinants). Whence the map $M \mapsto [\mathfrak{a}]$ is an isomorphism, and we are done.

14. Problem 14

We have the following commutative diagram with exact rows, which will referenced each part of this problem:

$$\begin{tikzcd}
 & M' \arrow[r, "\phi"] \arrow[d, "f"] & M \arrow[r, "\psi"] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r, "\phi'"] & N \arrow[r, "\psi'"] & N'' &
\end{tikzcd}$$

(a). Suppose that f and h are monomorphisms. Let $m \in \operatorname{Ker} g$. By commutativity, there exists $m' \in M'$ with $\phi(m') = m$. By commutativity,

$$\phi'(f(m')) = 0$$

Since ϕ' is injective by exactness, $f(m') = 0$.

But f is also injective, so that $m' = 0$ and $\phi(0) = 0 = m$, and g is also a monomorphism.

(b). Suppose that f and h are surjective. Let $n \in N$. Then, $\psi'(n) \operatorname{Im} \operatorname{Im} h$, since h is surjective, so there exists $m'' \in M''$ such that $h(m'') = \psi'(n)$.

By exactness, ψ is surjective, so there exists $m \in M$ such that $\psi(m) = m''$. By commutativity of the diagram, $\psi'(g(m)) = \psi(n)$, so that $g(m) - n \in \operatorname{Ker}\psi' = \operatorname{Im}\phi'$, so there exists $n' \in N'$ such that $\phi'(n') = g(m) - n$, and since f is surjective, there exists $m' \in M'$ such that $f(m') = n'$. By commutativity of the diagram,

$$g \circ \phi(m') = g(m) - n$$

$$\implies n = g(m - \phi(m')) \in \operatorname{Im} g$$

So that g is surjective.

(c). Assume $0 \to M' \to M$ and $N \to N'' \to 0$ are exact. By the Snake Lemma,

$$0 \longrightarrow \operatorname{Ker} f \longrightarrow \operatorname{Ker} g \longrightarrow \operatorname{Ker} h$$

$$\xrightarrow{\delta} \operatorname{Coker} f \longrightarrow \operatorname{Coker} g \longrightarrow \operatorname{Coker} h \longrightarrow 0$$

is also exact. However, we observe that if any of the above two kernels and cokernels vanish, so must the other. Hence the statement is a triviality.

15. Problem 15

The diagram that will be referenced in each part of this question is the following:

$$\begin{tikzcd}
M_1 \arrow[r, "a_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "a_2"] \arrow[d, "f_2"] & M_3 \arrow[r, "a_3"] \arrow[d, "f_3"] & M_4 \arrow[r, "a_4"] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r, "b_1"] & N_2 \arrow[r, "b_2"] & N_3 \arrow[r, "b_3"] & N_4 \arrow[r, "b_4"] & N_5
\end{tikzcd}$$

Note the above is commutative with exact rows. The format of the solutions will be a string of implications so as to make it very easy for the reader to follow along the diagram. Also, any element of its corresponding set will be denoted with the lower case letter with the same index (ie, $m_3 \in M_3$ always).

(a). We have:

$$\begin{aligned}
& m_3 \in \operatorname{Ker} f_3 \\
\implies & f_4(a_3(m_3)) = 0 \quad \text{(commutativity)} \\
\implies & a_3(m_3) = 0 \quad \text{(injectivity of } f_4\text{)} \\
\implies & m_3 \in \operatorname{Im} a_2 \quad \text{(exactness)} \\
\implies & a_2(m_2) = m_3 \quad \text{(by definition)} \\
\implies & b_2(f_2(m_2)) = 0 \quad \text{(commutativity)} \\
\implies & b_1(n_1) = f_2(m_2) \quad \text{(exactness)} \\
\implies & f_1(m_1) = n_1 \quad \text{(surjectivity of } f_1\text{)} \\
\implies & f_2(a_1(m_1)) = f_2(m_2) \quad \text{(commutativity)} \\
\implies & f_2(a_1(m_1) - m_2) = 0 \\
\implies & a_1(m_1) = m_2 \quad \text{(injectivity of } f_2\text{)} \\
\implies & m_2 \in \operatorname{Im} a_1 = \operatorname{Ker} a_2 \quad \text{(exactness)} \\
\implies & m_3 = a_2(m_2) = 0 \\
\implies & f_3 \text{ is injective}
\end{aligned}$$

(b). Employing the same convention as in part (a), we see

$$\begin{aligned}
&n_3 \operatorname{Im} N_3 \\
&\implies f_4(m_4) = b_3(n_3) \quad \text{(surjectivity of } f_4\text{)} \\
&\implies f_5(a_4(m_4)) \quad \text{(exactness and commutativity)} \\
&\implies a_4(m_4) = 0 \quad \text{(injectivity of } f_5\text{)} \\
&\implies a_3(m_3) = m_4 \quad \text{(commutativity)} \\
&\implies b_3(f_3(m_3) - n_3) = 0 \\
&\implies b_2(n_2) = f_3(m_3) - n_3 \quad \text{(exactness)} \\
&\implies f_2(m_2) = n_2 \quad \text{(surjectivity of } f_2\text{)} \\
&\implies f_3(a_2(m_2)) = f_3(m_3) - n_3 \quad \text{(commutativity)} \\
&\implies n_3 = f_3(m_3 - a_2(m_2))
\end{aligned}$$

So that f_3 is surjective.

16. Problem 16

Let $\{S_i, (f_{ji}\}_{i\in I}$ denote our inverse system, where each $f_{ji} : S_j \to S_i$ are all surjective. By simplicity, this implies that each f_{ji} is either trivial or an isomorphism.

If every $S_i = 1$, then we are done. Hence, suppose not. Given S_i, S_j, there exists k such that $k \geqslant i, j$. Then

$$S_k \cong S_i, \quad S_k \cong S_j$$

$$\implies S_i \cong S_j$$

Then any two nontrivial groups in our inverse system are necessarily isomorphic. Let S denote the common isomorphism. By assumption S is simple, it remains only to show that

$$\varprojlim S_i = S$$

The isomorphism is not so difficult to specify. Choose i such that S_i is nontrivial. The inclusion

$$\begin{aligned} I : & S \hookrightarrow \varprojlim S_i \\ & x \mapsto (x)_{i \in I} \end{aligned}$$

is injective, since any nonzero element must represent a nonzero class in some S_i. It remains to show surjectivity. Let $(x_i) \in \varprojlim S_i$. For every $j \leqslant i$, $f_{ji}(x_j) = x_i$, and for every $k \geqslant i$, $f_{ik}(x_i) = x_k$. Inverting yields $f_{ki}(x_k) = x_i$, so that every element is completely determined by the ith spot; whence $i(x_i) = (x_i)$, and we have an isomorphism.

17. Problem 17

(a). We have the inverse system

$$(\mathbb{Z}/p^n, \pi_{nm})$$

with

$$\pi_{nm}(a + (p^n)) = a + (p^m)$$

By definition, $\pi_{nn} \equiv \text{id}$. Now, set $\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n$. Let

$$\begin{aligned} \rho_n : & \mathbb{Z}_p \to \mathbb{Z}/p^n \\ & (a + (p^m))_{m \in \mathbb{N}} \mapsto a + (p^n) \end{aligned}$$

This is certainly surjective as any $m + (p^n)$ has preimage

$$(n + (p^n))_{n \in \mathbb{N}}$$

There are no zero divisors, since if k is a zero divisor, then $p^n | k$ for all $n \in \mathbb{N}$, which is possible only if $k = 0$.

The maximal ideal is merely $p\mathbb{Z}_p$, since one immediately sees that

$$\mathbb{Z}_p / p\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$$

which is, in particular, a field. This is also the unique maximal ideal since any other

$$x = (x_i + (p^n))_{n\in\mathbb{N}}$$

is pointwise invertible by merely noting that $\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}/p[[p]]$.

An element in the ring of formal power series is invertible if and only if the first term is a unit, which corresponds to elements $x \notin p\mathbb{Z}_p$, so that every element not contained in $p\mathbb{Z}_p$ is a unit, so this is maximal.

This also gives that p is the only prime in this ring, since any other prime would generate an idea contained in $p\mathbb{Z}_p$, and hence divide p.

Finally, for the UFD property, we can actually do better, since $\mathbb{Z}_p$ is a PID. To see this, merely note that every ideal must be an ideal in each entry, and ideals in every entry are principal. Hence, $\mathbb{Z}_p$ is principal, hence a UFD.

(b). By the Chinese Remainder theorem,

$$\mathbb{Z}/(a) \cong \bigoplus_i \mathbb{Z}/(p_i^{\alpha_i})$$

where $a = p_1^{\alpha_1} \dots p_k^{\alpha_k}$ is the prime factorization of a. Using this and the fact that inverse limits preserve direct products,

$$\begin{aligned} \varprojlim_{(a)} \mathbb{Z}/(a) &= \prod_{p \text{ prime}} \varprojlim_{n} \mathbb{Z}/(p^n) \\ &= \prod_{p \text{ prime}} \mathbb{Z}_p \end{aligned}$$

As asserted.

18. Problem 18

(a). The diagram

$$\begin{tikzcd} A_{n+1} \times M_{n+1} \arrow[r] \arrow[d] & M_{n+1} \arrow[d] \\ A_n \times M_n \arrow[r] & M_n \end{tikzcd}$$

commutes, so that

$$g_{n+1,n}(a_{n+1}m_{n+1}) = f_{n+1,n}(a_{n+1}) \cdot m_n$$

Then, let $\varprojlim A_n$ act on $\varprojlim M_n$ by

$$(a_n)_{n\in\mathbb{N}}(m_n)_{n\in\mathbb{N}} := (a_n m_n)_{n\in\mathbb{N}}$$

By the above commutative diagram, this action is well defined and preserves the structure of the inverse limit.

(b). Using part (a), we consider our maps. Observe that $M_n = A$ for each n, and, $A_n = /bbz/(p^n)$. Then, we have

$$\psi : A \to A$$

$$a \mapsto pa$$

$$\phi : \mathbb{Z}/(p^n) \to \mathbb{Z}/(p^{n+1})$$

$$m + (p^n) \mapsto m + (p^{n+1})$$

with trivial action

$$\mathbb{Z}/(p^n) \times A \to A$$

$$(m + (p^n), a) \mapsto ma$$

We then see that

$$(m + (p^n), a) \mapsto ma \mapsto pma$$

and

$$(m + (p^n), a) \mapsto (m + (p^{n+1}), pa) \mapsto pma$$

Whence the diagram in part (a) commutes, and using the result of (a), $\varprojlim A := T_p(A)$ is a module over $\varprojlim \mathbb{Z}/(p^n) = \mathbb{Z}_p$.

(c). We see that

$$(m + (p^n), a, b) \mapsto m(a, b) \mapsto pm(a, b)$$

and,

$$(m + (p^n), a, b) \mapsto (m + (p^{n+1}), pa, pb) \mapsto m(pa, pb)$$

And, since $pm(a, b) = m(pa, pb)$, the diagram in (a) commutes so the result follows immediately.

19. Problem 19

By definition, if $a \mapsto 0 \in \varinjlim A_n$, then $f_{ik}(a) = 0$ for some $k \geqslant i$.

20. Problem 20

Note that

$$\varinjlim_i \varinjlim_j A_{ij} \text{ and } \varinjlim_j \varinjlim_i A_{ij}$$

both satisfy the following universal property: for all $(i, j) \leqslant (r, s)$, there exist maps $f_{(i,j)}$ and $f_{(r,s)}$ for every

$$f_{(i,j),(r,s)} : A_{ij} \to A_{rs}$$

making the following commute:

$$\begin{tikzcd} & \varinjlim_i \varinjlim_j A_{ij} & \\ A_{ij} \arrow[ur, "f_{(i,j)}"] \arrow[rr, "f_{(i,j),(r,s)}"'] & & A_{rs} \arrow[ul, "f_{(r,s)}"'] \end{tikzcd}$$

$$\begin{tikzcd} & \varinjlim_j \varinjlim_i A_{ij} & \\ A_{ij} \arrow[ur, "f_{(i,j)}"] \arrow[rr, "f_{(i,j),(r,s)}"'] & & A_{rs} \arrow[ul, "f_{(r,s)}"'] \end{tikzcd}$$

And any other object satisfying the above must factor through the direct limits. Whence they factor through each other, and we have a natural isomorphism

$$\varinjlim_i \varinjlim_j A_{ij} \cong \varinjlim_j \varinjlim_i A_{ij}$$

Similarly, if we merely reverse the directions of the arrows in the above diagram, $\varprojlim_i \varprojlim_j A_{ij}$ and $\varprojlim_j \varprojlim_i A_{ij}$ also satisfy the same universal property, and are hence naturally isomorphic.

21. Problem 21

First, we need some notation. Elements of our direct limit can be written as classes $[M_i, m_i]$ with $m_i \in M_i$ and group operation

$$[M_i, m_i] + [M_j, m_j] := [M_k, \phi_{ik}(m_i) + \phi_{jk}(x_j)]$$

with $k \geqslant i, j$. By definition of direct limits, this is well defined. We also have induced maps u, v such that

$$u[M_i', m_i'] = [M_i, u_i(m_i')]$$

$$v[M_i, m_i] = [M_i'', v_i(m_i)]$$

where

$$0 \longrightarrow M_i' \xrightarrow{u_i} M_i \xrightarrow{v_i} M_i'' \longrightarrow 0$$

is exact for every i. Now we may prove exactness. Suppose first that $u([M_i, m_i]) = 0$. Then,

$$[M_i, u_i(m_i')] = 0 \implies f_{ij}(u_i(m_i')) = 0$$

for $j \geqslant i$. But,

$$f_{ij}(u_i(m_i')) = u_j(f_{ij}(m_i')) = 0$$

and since each u_j is a monomorphism, $f_{ij}(m_i') = 0$ for every $\geqslant i$, and we see that

$$[M_i', m_i'] = [M_i', 0]$$

so that u is a monomorphism. We also see:

$$\begin{aligned} vu([M_i', m_i']) &= v([M_i, u_i(m_i')]) \\ &= [M_i'', v_i u_i(m_i')] \\ &= [M_i'', 0] \end{aligned}$$

where the last equality follows from the fact that $\operatorname{Im} u_i \subset \operatorname{Ker} v_i$ for every i, and we thus deduce that $\operatorname{Im} u \subset \operatorname{Ker} v$. Let us consider the reverse inclusion now; suppose

$$\begin{aligned} v([M_i, m_i]) &= [M_i'', v_i(m_i)] \\ &= [M_i'', 0] \end{aligned}$$

Then, for all $j \geqslant i$, $g_{ij}(v_i(m_i)) = v_j(g_{ij}(u_i)) = 0$, so that given $g_{ij}(m_i) \in \operatorname{Ker} v_j$, there exists $m_j' \in M_j'$ such that $u_j(m_j') = g_{ij}(m_i)$, in which case

$$\begin{aligned} [M_i, m_i] &= [M_j, u_j(m_j')] \\ &= u([M_i', m_i']) \in \operatorname{Im} u \end{aligned}$$

So that $\operatorname{Ker} v = \operatorname{Im} u$. Finally, let $[M_i'', m_i''] \in \varinjlim M_i$. Then for each i, $u_i(m_i) = m_i''$ for some $m_i \in M_i$, so that

$$\begin{aligned} [M_i'', m_i''] &= M_i'', u_i(m_i)] \\ &= u([M_i, m_i]) \in \operatorname{Im} u \end{aligned}$$

And we conclude that

$$0 \longrightarrow \varinjlim M_i' \xrightarrow{v} \varinjlim M_i \xrightarrow{u} \varinjlim M_i'' \longrightarrow 0$$

is also exact.

22. Problem 22

(a). Consider the universal property of the direct sum. If we apply the contravariant functor $\text{Hom}(-, N)$, we reverse the direction of the inclusion maps in our universa property. We then get an induced map

$$u : \text{Hom}(\bigoplus_i M_i, N) \to \prod_i \text{Hom}(M_i, N)$$

We also get an inverse map

$$\begin{aligned}&\prod_i \text{Hom}(M_i, N) \to \text{Hom}(\bigoplus_i M_i, N)\\&(f_i) \mapsto f\end{aligned}$$

Where $f(m_i) = \sum_i f_i(m_i)$, $(m_i) \in \bigoplus_i M_i$. Whence,

$$\text{Hom}(\bigoplus_i M_i, N) \cong \prod_i \text{Hom}(M_i, N)$$

(b). We have a similar universal property. When we apply the covariant functor $\text{Hom}(N, -)$, we preserve the direction of our arrows and get an induced map

$$u : \text{Hom}(N, \prod_i M_i) \to \prod_i \text{Hom}(N, M_i)$$

And we have an inverse map

$$\begin{aligned}&\prod_i \text{Hom}(N, M_i) \to \text{Hom}(N, \prod_i M_i)\\&(f_i) \mapsto f\end{aligned}$$

where f is such that

$$f(n) = (f_i(n)) \in \prod_i M_i$$

and we conclude

$$\prod_i \text{Hom}(N, M_i) \cong \text{Hom}(N, \prod_i M_i)$$

23. Problem 23

We have the diagram

$$\begin{tikzcd} & \varprojlim_i M_i \arrow[dl] \arrow[dr] & \\ M_j \arrow[rr] & & M_i \end{tikzcd}$$

for $i \leqslant j$. Applying $\mathrm{Hom}(N, -)$, we have the induced diagram

$$\begin{tikzcd} & \mathrm{Hom}(N, \varprojlim_i M_i) \arrow[dl] \arrow[dr] & \\ \mathrm{Hom}(N, M_j) \arrow[rr] & & \mathrm{Hom}(N, M_i) \end{tikzcd}$$

So by the universal property, there exists a map

$$u : \mathrm{Hom}(N, \varprojlim_i M_i) \to \varprojlim_i \mathrm{Hom}(N, M_i)$$

And we construct an inverse

$$\begin{aligned} &\varprojlim_i \mathrm{Hom}(N, M_i) \to \mathrm{Hom}(N, \varprojlim_i M_i) \\ &[\mathrm{Hom}(N, M_i), f_i] \mapsto f \end{aligned}$$

where f is such that

$$f(n) = [M_i, f_i(n)]$$

whence

$$\mathrm{Hom}(N, \varprojlim_i M_i) \cong \varprojlim_i \mathrm{Hom}(N, M_i)$$

24. Problem 24

Let M be an R-module. Consider the set of finitely generated submodules of M, ordered by inclusion. We have the direct system

$$\{M_i, i_{ij}\}$$

where i_{ij} is the natural inclusion $M_i \hookrightarrow M_j$. We want to show that $M = \varinjlim M_i$. The map is naturally defined as

$$m \mapsto [M_i, m_i]$$

Now, if $m \mapsto [M_i, 0]$, then m is 0 is some finitely generated submodule of M, hence $m = 0$.

Now, given $[M_i, m_i] \in \varinjlim M_i$, note that $M = \bigcup_i M_i$, and we may take the preimage as $m_i \in M$ for any i. This is well defined, since if $i \leqslant j$, $i_{ij}(m_i) = m_j$, but $i_{ij}(m_i) = m_i$, merely viewed as an element of M_j. Hence,

$$M = \varinjlim M_i$$

25. Problem 25

We have an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

Consider the poset

$$S := \{(N, I) \mid |I| < \infty,\ N \subset K \cap R^I,\ N \text{ f.g}\}$$

Under the partial order

$$(N, I) \leqslant (N', I')$$

$$\iff N \leqslant N' \text{ and } I \subset I'$$

Now consider $\varinjlim R^I/N$. We want to show that this is isomorphic to F/K, as $F/K \cong M$. The map is trivial, we merey send

$$f + K \mapsto [R^I/N, f + N]$$

and by identical steps as in the previous problem, this is an isomorphism. For each R^I/N, we have the exact sequence

$$N \longrightarrow R^I \longrightarrow R^I/N \longrightarrow 0$$

As $N \subset K \cap R^I$, this is a finite presentation.

26. Problem 26

We first show this is a monomorphism. Consider

$$\begin{aligned}
&\varinjlim \operatorname{Hom}(E, M_i) \to \operatorname{Hom}(E, \varinjlim M_i) \\
&[\operatorname{Hom}(E, M_i), f_i] \mapsto f
\end{aligned}$$

where $f(n) = [M_i, f_i(n)]$. Suppose $[\operatorname{Hom}(E, M_i), f_i] \neq 0$, so $f_i \neq 0$ for some i. Then there exists $n \in E$ such that $f_i(n) \neq 0$, and injectivity follows by taking the contrapositive.

Suppose now that E is finitely generated and free, so that $E = R^n$ for some $n \in \mathbb{N}$. We then see

$$\begin{aligned}
\varinjlim \operatorname{Hom}(E, M_i) &= \varinjlim \operatorname{Hom}(R^n, M_i) \\
&= \varinjlim \Big(\operatorname{Hom}(E, M_i) \Big)^n \\
&= \varinjlim M_i^n \\
&= \Big(\varinjlim M_i \Big)^n
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{Hom}(E, \varinjlim M_i) &= \operatorname{Hom}(R^n, \varinjlim M_i) \\
&= \Big(\operatorname{Hom}(R, \varinjlim M_i) \Big)^n \\
&= \Big(\varinjlim M_i \Big)^n
\end{aligned}$$

So that these are indeed isomorphic in the free and finitely generated case. Suppose E is finitely presented and choose a presentation

$$F_0 \longrightarrow F_1 \longrightarrow E \longrightarrow 0$$

Apply the left exact contravariant functor $\mathrm{Hom}(-, M_i)$ and then the exact functor (by Problem 21) $\varinjlim$ to get the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \varinjlim \mathrm{Hom}(E, M_i) \arrow[r] \arrow[d] & \varinjlim \mathrm{Hom}(F_0, M_i) \arrow[r] \arrow[d] & \varinjlim \mathrm{Hom}(F_1, M_i) \arrow[d] \\
0 \arrow[r] & \mathrm{Hom}(E, \varinjlim M_i) \arrow[r] & \mathrm{Hom}(F_0, \varinjlim M_i) \arrow[r] & \mathrm{Hom}(F_1, \varinjlim M_i)
\end{tikzcd}$$

Where the vertical arrows are the natural maps. Then, using exactness we easily deduce that first vertical arrow must be a surjection, since we have already shown that the second 2 are. Whence we have an isomorphism

$$\varinjlim \mathrm{Hom}(E, M_i) \cong \mathrm{Hom}(E, \varinjlim M_i)$$

27. Problem 27

Define the product

$$(x + A_{n-1})(y + A_{m-1}) := xy + A_{n+m-1}$$

This is well defined and preserves the graded structure as $xy \in A_{n+m}$, so

$$\frac{A_n}{A_{n-1}} \cdot \frac{A_m}{A_{m-1}} \subset \frac{A_{n+m}}{A_{n+m-1}}$$

And this is the multiplication rule for the associated graded module $\mathrm{gr}(A)$.

28. Problem 28

(a). We have the natural definition

$$\begin{aligned}
\mathrm{gr}_i(L) :&\mathrm{gr}_i \to \mathrm{gr}_i(B) \\
&a + A_{i-1} \mapsto L(a) + B_{i-1}
\end{aligned}$$

Let us show this is well defined. Suppose that $a + A_{i-1} = a' + A_{i-1}$. Then $a - a' \in A_{i-1}$, so that

$$L(a - a') \in B_{i-1} \implies L(a) + B_{i-1} = L(a') + B_{i-1}$$

So this is well defined.

(b). Let $b \in B_i$, and without loss of generality assume that $b \notin B_{i-1}$. Since $\mathrm{gr}_i(L)$ is an isomorphism, there exists $a_0 \in A_i$ such that $b - L(a_0) \in B_{i-1}$. Similarly, we may find $a_1 \in A_{i-1}$ such that

$$(b - L(a_0)) - L(a_1) \in B_{i-2}$$

Iterating this, after $i + 1$ times we have found $a_k \in A_{i-k}$ such that

$$b - \sum_{k=0}^{i} L(a_k) \in B_{-1} = \{0\}$$

Whence $b - \sum_{k=0}^{i} L(a_k) = 0$, implying that

$$b = L\Big(\sum_{k=0}^{i} a_k\Big)$$

so that L is surjective.

Suppose now that $L(a) = 0$ for $a \in A$. Then, $a \in A_i$ for some i, and since $\mathrm{gr}_i(L)$ is an isomorphism, $a \in A_{i-1}$. Iterating this, we see that $a \in A_j$ for all $j \leqslant i$, and in particular, $a \in A_{-1} = \{0\}$, so that $a = 0$, and L is an isomorphism.

29. Problem 29

(a). These are algebras just by definition, and indeed we see that $\det(N - \lambda I) = \lambda^n$ for $N \in \mathfrak{n}_i$, whence $N^n = 0$.

(b). Closure follows from

$$(I+X)(I+Y) = I + X + Y + XY$$

Since $\mathfrak{n}$ is an algebra, this remains in our set. Associativity follows from associativity of matrix multiplication. Lastly, $I = I + 0$ is the identity element.

Finally, suppose that X is nilpotent of degree i; we have

$$(I+X)(I - X + X^2 - \cdots + (-1)^{i-1}X^{i-1}) = I - X^i = I$$

So all elements are invertible, and we have a group.

(c). Note that exp is a polynomial function, where

$$\exp(X) = \sum_{n=0}^{\infty} \frac{X^n}{n!}$$

The sum is not actually infinite since X is nilpotent, so we have a polynmial function. To show this is a bijection, we only need show that log is the inverse. We see:

$$\begin{aligned}
\log(\exp(X)) &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \big(\exp(X) - I\big)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \cdot \exp(kX) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \cdot \sum_{m=0}^{\infty} \frac{X^m}{m!} \cdot k^m \\
&= -\sum_{m=0}^{\infty} \frac{X^m}{m!} \Big(\sum_{n=1}^{m} \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m \\
&+ \sum_{n=m+1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m \Big)
\end{aligned}$$

Now, consider

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m x^k$$

This is the resulting of applying the operator $\left(x\frac{d}{dx}\right)^m$ so $(1-x)^n$. When $n < m$, we see that the end result will still have a factor of $1 - x$, so that, setting $x = 1$, whenever $m < n$ we have

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k k^m = 0$$

And the above sum becomes

$$\begin{aligned}
&-\sum_{m=0}^{\infty} \frac{X^m}{m!} \sum_{n=1}^{m} \frac{1}{n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} k^m \\
&= -\sum_{m=0}^{\infty} \frac{X^m}{m!} \sum_{n=1}^{m} \sum_{k=0}^{n} (-1)^k \frac{1}{k} \binom{n-1}{k-1} k^m \\
&= -\sum_{m=0}^{\infty} \frac{X^m}{m!} \sum_{k=1}^{m} (-1)^k \cdot k^{m-1} \sum_{n=k}^{m} \binom{n-1}{k-1}
\end{aligned}$$

Now consider

$$\sum_{n=k}^{m} \binom{n-1}{k-1}$$

This is the coefficient of x^{k-1} in the sum

$$(1+x)^{k-1} + (1+x)^k + \cdots + (1+x)^{m-1}$$

which, by the geometric sum formula, is just equal to

$$\frac{(1+x)^{m-k+1}}{x} - \frac{(1+x)^{k-1}}{x}$$

This has no degree $k-1$ terms, but for the form before that, x^{k-1} has coefficient

$$\binom{m-k}{k} = \binom{m}{k}$$

So we find

$$\sum_{k=n}^{m} \binom{n-1}{k-1} = \binom{m}{k}$$

and our sum becomes

$$-\sum_{m=0}^{\infty} \frac{X^m}{m!} \sum_{k=1}^{m} (-1)^k \cdot k^{m-1} \binom{m}{k}$$

Now, when $m > 1$, we have already shown above that

$$\sum_{k=1}^{m} (-1)^k \cdot k^{m-1} \binom{m}{k} = \sum_{k=0}^{m} (-1)^k \cdot k^{m-1} \binom{m}{k} = 0$$

and, when $m = 1$,

$$\sum_{k=1}^{1} (-1)^k \binom{1}{k} = -1$$

So that all terms of order $m > 1$ vanish, and we are merely left with X. Hence, $\log(\exp(X)) = X$. Also by rearranging the terms in our series, we also see that

$$\log(\exp(X)) = \exp(\log(X)) = X$$

so this log is a left and right inverse, giving that exp is indeed a bijection, as desired.